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On the multi-component extensions of the Burgers equation

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Abstract. We construct a multi-component Burgers equation by means of a hereditary symmetry. This n-component system has n parameters.

It is an interesting and important problem to construct multi-component analogies or coupled systems for the well known integrable equations. The point is that by this study, on the one hand, one may enlarge the class of integrable equations and furthermore obtain some systems which may be physically significant. On the other hand, such integrable coupling may lead to a better understanding of integrability.

Thus far, this problem has been examined by many authors and different approaches have been taken. For the celebrated KDV systems, a number of coupled systems are constructed. Among these equations, we cite the well known Hirota–Satsuma system [1] and Ito equation [2]. Very recently, Antonowicz and Fordy [3] introduced coupled KDV and coupled Harry Dym systems with n components. Their method is extended to Boussinesq and Toda cases so that coupled Boussinesq systems and coupled Toda equation are constructed [4, 5]. We notice that the systems studied by this method are all S-integrable, that is, the equation can be solved by inverse scattering transformation.

Besides the S-integrable systems, there exist so-called C-integrable systems. Among many of the systems, the Burgers equation is the best known. Recently, Calogero and his collaborators studied this type of equation intensively [6, 7]. We also notice that Mikhailov *et al* [8] classified integrable systems by seeking higher symmetries or generalized symmetries. In particular, they provide some extensions of the Burgers equation. In this paper we generalize the approach of Antonowicz and Fordy to the Burgers system. The first step in this direction was taken recently by *Ma*. In [9] he started with a spectral problem and eventually obtained a coupled Burgers equation. We will directly construct a family of coupled Burgers systems without the use of a spectral problem. It turns out that our systems is very general and contains the systems in [9] as a special case.

We start with the following matrix operator

$$\boldsymbol{R} = \begin{bmatrix} 0 & 0 & \cdots & J_0 \\ 1 & 0 & \cdots & J_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & J_{n-1} \end{bmatrix}. \tag{1}$$

This operator first appeared in the work of Antonowicz and Fordy [3] when they considered the coupled KDV and Harry Dym systems. The operator is always a recursion operator for their systems. We should point out that this operator seems a universal form and it appeared many times.

Now let us specify the elements J_i in the following way:

$$J_i = 2\epsilon_i \partial + u_i + u_{i_x}\partial^{-1} \qquad i = 0, \ldots, n-1 \tag{2}$$

where ϵ_i are the constants and u_i are the field variables. For this matrix integro-differential operator, our main observation is:

Theorem. The operator $\boldsymbol{R}$ is hereditary with J_i defined by (2).

Proof. A direct calculation.

With hereditary symmetry in hand, we now construct a hierarchy of integrable systems. To this end, it is easy to prove that $\boldsymbol{R}$ is a strong symmetry of the translation flows: $\boldsymbol{u}_t = \boldsymbol{u}_x$. Thus, we have:

$$\boldsymbol{u}_{t_{m+1}} = \boldsymbol{R}^m \boldsymbol{u}_x. \tag{3}$$

where $\boldsymbol{u} = (u_0, \ldots, u_{n-1})^{\mathrm{T}}$.

According to the standard theory [10, 11], each flow in (3) has infinite symmetries which commute.

Now, let us write down the first non-trivial flow:

$$\boldsymbol{u}_{t_2} = \begin{bmatrix} 2\epsilon_0 u_{n-1_{xx}} + (u_0 u_{n-1})_x \\ u_{0_x} + 2\epsilon_1 u_{n-1_x} + (u_1 u_{n-1})_x \\ \vdots \\ u_{n-2_x} + 2\epsilon_{n-1} u_{n-1_{xx}} + 2u_1 u_{n-1_x} \end{bmatrix}. \tag{4}$$

When $\epsilon_i = 0$, $i = 0, \ldots, n-2$ and $\epsilon_{n-1} = \frac{1}{4}$, we recover the systems proposed in [9] from (3). This flow reduces to the Burgers equation when the first components are absent. In passing, we may remark that the systems in [9] are somewhat analogous to the Ito type of the extensions of the Burgers equation.

Let us take a close look at the simplest non-trivial case with $\epsilon_0 = \frac{1}{2}$ and $\epsilon_1 = 0$: the two components case,

$$u_{0_{t_2}} = u_{1_{xx}} + (u_0 u_1)_x \qquad u_{1_{t_2}} = u_{0_x} + 2u_1 u_{1_x} \tag{5}$$

and

$$u_{0_{t_3}} = (\partial^2 + \partial u_0)(u_0 + u_1^2) \qquad u_{1_{t_3}} = u_{1_{xx}} + 2(u_0 u_1)_x + 3u_1^2 u_{1_x}. \tag{6}$$

It is interesting to see that this time t_3 flow (6) can be reduced to the Burgers equation.

The important question is now if the system, say equation (5), is or is not integrable. Since the systems are unlikely to be Hamiltonian, the integrability in the Hamiltonian sense seems not to be possible. One might think that there exists a Cole–Hopf transformation in this context, but I failed to find it. Thus far, we just know that each flow of the hierarchy (3) admits infinite generalized symmetries. The systems are integrable in this sense. It will be interesting to see if the systems share any other properties of integrable systems.

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